

COMPLETE INTERSECTION VARIETIES WITH AMPLE COTANGENT BUNDLES

by

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Abstract. — Any smooth projective variety contains many complete intersection subvarieties with ample cotangent bundles, of each dimension up to half its own dimension.

1. Introduction

It is expected that projective varieties with ample cotangent bundles should be abundant, at least under reasonable assumptions; in [Sch92], Schneider proved that such a variety cannot be embedded in projective space in a way that its dimension is larger than its codimension. Yet, not so many examples were known until recently. Looking for such examples and generalizing a question of Schneider in the cited work, one can wonder if a smooth N -dimensional projective variety contains subvarieties with ample cotangent bundles (besides curves). In this paper, we answer this question and establish that, after taking into account Schneider’s condition, subvarieties with ample cotangent bundles are ubiquitous.

Main result. — *In any smooth projective variety M , for each $n \leq \dim(M)/2$, there exists a smooth subvariety of dimension n with ample cotangent bundle.*

To give some illustrative examples: in view of the aforementioned Schneider’s result, this statement is sharp for $M = \mathbb{P}^N$ (which has anti-ample cotangent bundle), by [Deb05] it is also sharp if M is an abelian variety (which has trivial cotangent bundle), and the statement becomes obviously non-sharp (and trivial) if M itself has ample cotangent bundle.

In an ambient variety M , a natural way to construct subvarieties with ample cotangent bundles is to consider complete intersections of very ample divisors. Indeed, there are several ways to see that cotangent bundles of smooth hypersurfaces carry more “positivity” than the cotangent bundle of the original variety itself (consider *e.g.* adjunction formula, or observe that cotangent bundles of hypersurfaces are quotients of the cotangent bundle of the ambient variety). Taking the complete intersection of more and more hypersurfaces, it is a natural question to ask at which point the cotangent bundle may become ample. We say that a property holds for a *general* member of some family, if it holds for any member of the family over a dense Zariski-open subset of the base. In [Deb05], Debarre conjectured that complete intersections in projective space having codimensions at least equal to their dimensions should have ample cotangent bundles, provided the hypersurfaces one intersects are general and sufficiently ample. This conjecture was motivated by his proof of the analogous statement for complete intersections in abelian varieties (see also [Deb13, Ben11]).

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In the spirit of this conjecture, we prove the following statement, which implies our main result.

Theorem 1.1. — *On a N -dimensional smooth projective variety M , equipped with a very ample line bundle $\mathcal{O}_M(1)$, for codimensions $N/2 \leq c \leq N$, for each $\delta = (\delta_1, \dots, \delta_c) \in (\mathbb{N}^*)^c$ and any sufficiently large multiple*

$$(d_1, \dots, d_c) = \nu \cdot (\delta_1, \dots, \delta_c) \in (\mathbb{Q}_+ \cdot \delta) \cap (\mathbb{N}^*)^c \quad \text{with } \nu \geq \nu(\delta)$$

the complete intersection $X := H_1 \cap \dots \cap H_c$ of general hypersurfaces $H_1 \in |\mathcal{O}(d_1)|, \dots, H_c \in |\mathcal{O}(d_c)|$ has ample cotangent bundle.

During the last steps of the preparation of this paper, we were informed that Xie announced a proof of ampleness of the cotangent bundles of general complete intersections in \mathbb{P}^N , with codimensions at least as large as their dimensions, with a uniform lower bound on the degrees $d_i \geq N^{N^2}$ ([Xie15]). In the case $M = \mathbb{P}^N$, his result is thus stronger than Theorem 1.1, concerning the hypothesis on the degrees.

Since ampleness is an open condition in families, to prove Theorem 1.1 for one fixed multi-degree d_1, \dots, d_c , it suffices to show the existence of an example complete intersection $H_1 \cap \dots \cap H_c$ with ample cotangent bundle and the required multi-degree. Our construction of such examples (see Section 2.1) has two inherent flaws, that are equally inconsequential regarding our main result. The first one is that we cannot use powers of different very ample divisors to define H_1, \dots, H_c . The second one is that the lower bound on the degrees is not uniform (namely it depends on the direction).

Let us mention that a factorization trick due to Xie (pp. 21–23 in [Xie15]) could be adapted in our situation to give a uniform lower bound on the degrees, modulo a technical generalization of our arguments.

Our result is in the vein of several previous ones we would like to mention. In an unpublished work, Bogomolov constructed complete intersection varieties with ample cotangent bundles in products of varieties with big cotangent bundles (we refer to [Deb05] for a treatment of this result). We have already mentioned that Debarre proved in [Deb05] that in an abelian variety, a general complete intersection of sufficiently ample divisors, whose codimension is at least as large as its dimension, has ample cotangent bundle. In [Bro14], the first named author proved, among other results in the direction of the quoted conjecture of Debarre, that Theorem 1.1 (with an effective bound on the degrees) holds for complete intersection surfaces in projective space, and in [Bro15] he also proved that a general complete intersection X of multi-degree (d, \dots, d) in \mathbb{P}^N , with $\text{codim } X \geq 3 \dim X - 2$, has ample cotangent bundle when $d \geq 2N + 3$.

The present work finds its roots in [Bro15], and pushes further the ideas initiated there to complete the study of families of complete intersection varieties, here in a fully geometric way. Although our intuitions grew from the geometric interpretation of the cohomological computations arising in [Bro15], to make this approach effective is not a goal of the present text. We rather choose to emphasize the intrinsic simplicity and the geometric nature of the proof.

Let us now outline how we establish Theorem 1.1. The proof relies on two ideas, exploited as follows.

A polynomial of high degree and its differential are (generically) independent. — This idea was already very present in Debarre’s proof in abelian varieties [Deb05], as well as in the previous work of the first named author. Inspired by the treatment of families defined by Fermat-type hypersurfaces in [Bro15], we consider the family $\mathcal{X} \rightarrow S$ of complete intersection subvarieties in M cut out by certain bihomogeneous sections (see Section 2.1). For these sections, we show that besides being independent, the differentials furthermore have the same shape as the original sections. Formally, one obtains $2c \geq N$ sections of the same type. After making this idea rigorous, we are able to define a map \mathcal{P} from the projectivized⁽¹⁾ relative cotangent bundle $\mathbb{P}(\Omega_{\mathcal{X}/S})$ to a certain family $\mathcal{Y} \rightarrow \mathbf{G}$ of subschemes in \mathbb{P}^N defined by $2c$ universal equations (see Section 2.2). Since $2c \geq N$, the general fiber of this family has dimension ≤ 0 .

Use the positivity of the parameter space of the family. — This idea was already present in Voisin’s variational method ([Voi96]). However, instead of pulling back the positivity from the parameter space of the family \mathcal{X} under consideration, we rather use the map \mathcal{P} to pull back the positivity from the family \mathcal{Y} ; which is itself easily deduced from the positivity on \mathbf{G} , since the general fibers of $\mathcal{Y} \rightarrow \mathbf{G}$ are finite. To the best of our knowledge, this approach is new.

To effectively pull-back the positivity, the central idea is that the images by \mathcal{P} of curves lying in general fibers of $\mathbb{P}(\Omega_{\mathcal{X}/S})$ should avoid the locus of positive dimensional fibers of $\mathcal{Y} \rightarrow \mathbf{G}$. We establish this fact thanks to a result of Benoist [Ben11] which yields that this locus is small. See Section 2.3, 2.4 for full detail.

⁽¹⁾ for a vector bundle E on a variety X we denote by $\pi_E: \mathbb{P}(E) \rightarrow X$ the projectivization of rank one quotients of E

2. Ampleness of the cotangent bundle of general complete intersections

We work over an algebraically closed field k of characteristic 0. Recall that M is an N -dimensional smooth projective variety over k , equipped with a very ample line bundle $\mathcal{O}_M(1)$.

Throughout this text we use the following notation. We denote by $\{|J| = \delta\} \subset \mathbb{N}^{N+1}$ the subset of multi-integers $J = (J_0, \dots, J_N)$ such that $|J| := J_0 + J_1 + \dots + J_N = \delta$. The support of a multi-index J is denoted

$$[J] := \{i \in \{0, 1, \dots, N\} \mid J_i \neq 0\}.$$

For $\emptyset \subsetneq I \subseteq \{0, \dots, N\}$ with $\#I = (1 + k)$, there are $\binom{k+\delta}{k}$ multi-indices of length δ with $[J] \subset I$. These parametrize non-vanishing degree δ homogeneous monomials on

$$k^I := \{(Z_0, \dots, Z_N) \in k^{N+1} \mid Z_i = 0, \forall i \notin I\}.$$

When we work on a chart V in a trivializing affine covering \mathfrak{B} for $\mathcal{O}_M(1)$, for $\sigma \in H^0(M, \mathcal{O}_M(1))$ we will denote by $(\sigma)_V \in \mathcal{O}(V)$ the regular function corresponding to σ under that trivialization.

2.1. Complete intersections cut out by bihomogeneous sections. — We fix $N + 1$ sections in general position, $\zeta_0, \dots, \zeta_N \in H^0(M, \mathcal{O}_M(1))$, and we set $D_i := (\zeta_i = 0)$ for $i = 0, \dots, N$. By “general position”, we mean that each of the D_i is smooth, and that the divisor $D = \sum_i D_i$ is simple normal crossing.

For a codimension c satisfying the hypothesis $N/2 \leq c \leq N$ of Theorem 1.1, and two c -tuples of positive integers $\varepsilon = (\varepsilon_1, \dots, \varepsilon_c)$, $\delta = (\delta_1, \dots, \delta_c)$, we construct the family \mathcal{X} mentioned in the introduction as follows. For $p = 1, \dots, c$, for $\mathbf{a}^p = (a_j^p \in H^0(M, \mathcal{O}_M(\varepsilon_p)))_{|J|=\delta_p}$, and for a positive integer r fixed later according to our needs, we consider the bihomogeneous section of $\mathcal{O}_M(\varepsilon_p + (r+1)\delta_p)$ over M defined by⁽²⁾

$$(1) \quad E^p(\mathbf{a}^p, \cdot): x \mapsto \sum_{|J|=\delta_p} a_J^p(x) \zeta(x)^{(r+1)J}.$$

We rather want to let \mathbf{a}^p vary in the parameter space $S_p := H^0(M, \mathcal{O}_M(\varepsilon_p))^{\binom{N+\delta_p}{N}}$ and to think at E^p as a section $E^p \in H^0(S_p \times M, \mathcal{O}_M(d_p))$ where $d_p := \varepsilon_p + (r+1)\delta_p$.

We then consider the family $\overline{\mathcal{X}} \subset S_1 \times \dots \times S_c \times M$ of complete intersection varieties in M defined by those sections, *i.e.*

$$\overline{\mathcal{X}} := \{(\mathbf{a}^1, \dots, \mathbf{a}^c; x) \in S_1 \times \dots \times S_c \times M \mid E^1(\mathbf{a}^1, x) = 0, \dots, E^c(\mathbf{a}^c, x) = 0\}.$$

Fact 2.1. — *The family $\overline{\mathcal{X}} \rightarrow S_1 \times \dots \times S_c$ is generically smooth.*

Proof. — To see this, it suffices to produce one smooth member of this family. This is a straightforward induction based on the following statement: *Given $\varepsilon, \delta > 0$ and a smooth subvariety X of M there exists a section $\sigma = \sum_{i=0}^{N+1} a_i \zeta_i^{(r+1)\delta} \in H^0(M, \mathcal{O}_M(\varepsilon + (r+1)\delta))$, where $a_i \in H^0(M, \mathcal{O}_M(\varepsilon))$, such that $X \cap (\sigma = 0)$ is smooth.* The proof is just a minor modification of the proof of the classical Bertini theorem, we sketch it here for the sake of completeness. Consider the set $\Sigma \subset H^0(M, \mathcal{O}_M(\varepsilon))^{N+1} \times X$ composed of element of the form (a_0, \dots, a_N, x) such that the hypersurface H_a defined by $(\sigma_a := \sum_{i=0}^{N+1} a_i \zeta_i^{(r+1)\delta} = 0)$ contains x but $X \cap H_a$ has a singularity at x . Set $H^0(M, \mathcal{O}_M(\varepsilon))_x^{N+1} = \{\mathbf{a} = (a_0, \dots, a_N) \mid x \in H_a\}$, this is a hyperplane of $H^0(M, \mathcal{O}_M(\varepsilon))^{N+1}$. For a fixed x we are going to estimate the dimension of $\Sigma_x := \text{pr}_2^{-1}(x)$. Fix an affine neighborhood $V \subset M$ of x , where we trivialize $\mathcal{O}_M(1)$. Then we have a linear map

$$H^0(M, \mathcal{O}_M(\varepsilon))_x^{N+1} \xrightarrow{\varphi_x} \Omega_{X \cap V, x},$$

defined by $\varphi_x(\mathbf{a}) = d(\sigma_a)_V|_{\Omega_{X,x}}$. Observe that $d(\sigma_a)_V \in \Gamma(V, \Omega_V)$, so it makes sense to restrict it to $\Omega_{X \cap V, x}$. Observe that $\Sigma_x \cong \ker \varphi_x$. Let us prove that φ_x is surjective, to see this it suffices to prove that for $\xi \in T_x X$, there exists \mathbf{a} such that $\varphi_x(\mathbf{a})(\xi) \neq 0$. Fixing $\xi \in T_x X$, it suffices to consider $0 \leq i_0 \leq N$, such that $\zeta_{i_0}(x) \neq 0$, and to take the a_i 's such that $a_i(x) = 0$ for any i , $d(a_{i_0})_V|_x(\xi) \neq 0$ and $d(a_i)_V|_x(\xi) = 0$ otherwise (this is possible since $\mathcal{O}_M(1)$ is very ample). A direct computation then shows that $\varphi_x(\mathbf{a})(\xi) = d(a_{i_0})_V|_x(\xi) \zeta_{i_0}^{(r+1)\delta}(x) \neq 0$. This shows the desired surjectivity. Therefore,

$$\dim \Sigma_x = h^0(M, \mathcal{O}_M(\varepsilon))^{N+1} - 1 - \text{rank } \varphi_x = h^0(M, \mathcal{O}_M(\varepsilon))^{N+1} - \dim X - 1.$$

From which we infer that $\dim \Sigma < h^0(M, \mathcal{O}_M(\varepsilon))^{N+1}$. This implies that Σ does not dominate $H^0(M, \mathcal{O}_M(\varepsilon))^{N+1}$, whence the result. \square

⁽²⁾here and throughout the text, we use the standard multi-index notation for multivariate monomials

We will from now on restrict ourselves to the dense open subset $S \subset S_1 \times \cdots \times S_c$ parametrizing smooth varieties, and we will prove the ampleness of the cotangent bundle of the general members of the universal family

$$\mathcal{X} := \{(\mathbf{a}^1, \dots, \mathbf{a}^c; x) \in S \times M \mid E^1(\mathbf{a}^1, x) = 0, \dots, E^c(\mathbf{a}^c, x) = 0\}.$$

We denote pr_1 and pr_2 the natural projections from \mathcal{X} to the factors of the product $S \times M$.

After projectivization, the relative cotangent sheaf $\Omega_{\mathcal{X}/S}$ of the family \mathcal{X} gives rise to the family we are interested in, namely

$$\mathcal{X}^{[1]} := \mathbb{P}(\Omega_{\mathcal{X}/S}) \subset S \times \mathbb{P}(\Omega_M).$$

We denote $\text{pr}_1^{[1]}$ and $\text{pr}_2^{[1]}$ the natural projections from $\mathcal{X}^{[1]}$ to the factors of the product $S \times \mathbb{P}(\Omega_M)$. A fiber $(\text{pr}_1^{[1]})^{-1}(\mathbf{a}^\bullet)$ of $\mathcal{X}^{[1]} \rightarrow S$ projects onto $X_{\mathbf{a}^\bullet} := (\text{pr}_1)^{-1}(\mathbf{a}^\bullet)$ and coincide with $\mathbb{P}(\Omega_{X_{\mathbf{a}^\bullet}})$.

As mentioned in the introduction, the motivation for the choice of this class of examples comes from the previous work [Bro15] by the first author (where $\delta_p = 1$). See this text for comments on what were there called “Fermat-type” hypersurfaces and the interesting properties these share with our kind of equations.

2.2. Construction of families of non-positive-dimensional subschemes. — We work temporarily on a chart V in a trivializing affine covering \mathfrak{B} for $\mathcal{O}_M(1)$. The family $\mathcal{X}^{[1]} \subset S \times \mathbb{P}(\Omega_M)$ is locally defined by the equations $E^1, dE^1, \dots, E^c, dE^c$.

The main geometric idea in our proof is the construction of a family \mathcal{Y} of non-positive-dimensional subschemes defined by universal equations, together with map $\mathcal{Y} : \mathcal{X}^{[1]} \rightarrow \mathcal{Y}$ used to “pullback positivity”. We get a number of equations that outreaches the dimension if we use both the equations E^p and their differentials dE^p —recall that $2c \geq N$ —. We thus think at E^p and dE^p as two independent polynomials of degree δ_p in the variables $\zeta_0^r, \dots, \zeta_N^r$. Namely, for $\mathbf{a}^p \in S_p$ we consider locally⁽³⁾

$$\begin{aligned} (E^p(\mathbf{a}^p, \cdot))_V &= \sum_{|J|=\delta_p} (\alpha_J^p(\mathbf{a}^p, \cdot))_V (\zeta_V^J)^r \in \mathcal{O}(V); \\ d(E^p(\mathbf{a}^p, \cdot))_V &= \sum_{|J|=\delta_p} (\theta_J^p(\mathbf{a}^p, \cdot))_V (\zeta_V^J)^r \in H^0(V, \Omega_V); \end{aligned}$$

where $\alpha_J^p(\mathbf{a}^p, \cdot) := a_J^p \zeta^J$ and where

$$(2) \quad (\theta_J^p(\mathbf{a}^p, \cdot))_V := (\zeta^J)_V d(a_J^p)_V + (r+1)(a_J^p)_V d(\zeta^J)_V \in H^0(V, \Omega_V).$$

The local sections θ_J^p will soon disappear in favour of global sections that these will help to construct.

Actually, for each $p = 1, \dots, c$, for each $\mathbf{a}^p \in S_p$ and $\xi \in \mathbb{P}(\Omega_M)$, although the two points $(\alpha_J^p(\mathbf{a}^p, \xi))_{|J|=\delta_p}$ and $(\theta_J^p(\mathbf{a}^p, \xi))_{|J|=\delta_p}$ in $k^{\binom{N+\delta_p}{N}}$ depend on the choice of trivialization, the vector space $\Delta_p(\mathbf{a}^p, \xi)$ spanned by them does not. Under the assumption that this space is two dimensional, we get a well defined element in the Grassmannian of 2-planes $\text{Gr}_2(k^{\binom{N+\delta_p}{N}})$. Considering the Plücker embedding $\text{Gr}_2(k^{\binom{N+\delta_p}{N}}) \rightarrow \mathbb{P}(\Lambda^2 k^{\binom{N+\delta_p}{N}})$, one is naturally led to define locally the (relative) differential form

$$(3) \quad (\beta_{J_1, J_2}^p)_V := \det \begin{pmatrix} (\alpha_{J_1}^p)_V & (\alpha_{J_2}^p)_V \\ (\theta_{J_1}^p)_V & (\theta_{J_2}^p)_V \end{pmatrix} \in H^0(S_p \times V, \Omega_V).$$

Now, it is a standard exercise to show that these local sections $(\beta_{J_1, J_2}^p)_V$ glue together as a global section β_{J_1, J_2}^p . Indeed, one can classically define *Wrońskian differential forms*:

Lemma 2.2. — *Let $L \rightarrow X$ be a line bundle over a smooth variety X , and let $\alpha_1, \alpha_2 \in H^0(X, L)$ be global sections. Whereas $d\alpha_i$ may not be a global section of the twisted cotangent bundle $\Omega_X \otimes L$, the Wrońskian section*

$$\det \begin{pmatrix} \alpha_1 & \alpha_2 \\ d\alpha_1 & d\alpha_2 \end{pmatrix}$$

is always a global section in $H^0(X, \Omega_X \otimes L^{\otimes 2}) \cong H^0(\mathbb{P}(\Omega_X), \mathcal{O}_{\mathbb{P}(\Omega_X)}(1) \otimes \pi_{\Omega_X}^ L^{\otimes 2})$.*

Proof. — The quotient α_1/α_2 being a rational function on X , we can consider its logarithmic differential, which is a logarithmic form along the divisor $(\alpha_1\alpha_2 = 0)$. Multiplying by $\alpha_1\alpha_2$ cancels out the poles, and yields a section in $H^0(X, \Omega_X \otimes L^2)$. A short computation shows that this is precisely the Wrońskian form of the statement. \square

⁽³⁾ to evaluate sections defined on M at points of $\mathbb{P}(\Omega_M)$, we only imply the pre-composition by the projection $\pi_{\Omega_M} : \mathbb{P}(\Omega_M) \rightarrow M$

Next, it is not hard to see that $\zeta^{rJ_1+rJ_2}\beta_{J_1,J_2}^p \in H^0(S_p \times \mathbb{P}(\Omega_M), \mathbb{L}(2d_p))$ is the Wronskian section associated to the global sections $a_{J_1}^p \zeta^{(r+1)J_1}$ and $a_{J_2}^p \zeta^{(r+1)J_2}$ in $H^0(S_p \times M, \mathcal{O}_M(d_p))$. For more fluidity, we denote

$$\mathbb{L}(a) := \mathcal{O}_{\mathbb{P}(\Omega_M)}(1) \otimes (\pi_{\Omega_M})^* \mathcal{O}_M(a), \quad (\pi_{\Omega_M}: \mathbb{P}(\Omega_M) \rightarrow M).$$

The quotient of $\zeta^{rJ_1+rJ_2}\beta_{J_1,J_2}^p$ by $\zeta^{rJ_1+rJ_2}$ yields a rational section; since it has clearly no poles, we obtain

$$(4) \quad \beta_{J_1,J_2}^p \in H^0(S_p \times \mathbb{P}(\Omega_M), \mathbb{L}(2d_p - 2r\delta_p)) = H^0(S_p \times \mathbb{P}(\Omega_M), \mathbb{L}(2\varepsilon_p + 2\delta_p)).$$

These sections give rise to a rational map $\Delta^p: S \times \mathbb{P}(\Omega_M) \dashrightarrow \text{Gr}_2(\mathbb{k}^{\binom{N+\delta_p}{N}})$ given by

$$\Delta^p(\mathbf{a}^p, \xi) = \left[\beta_{J_1,J_2}^p(\mathbf{a}^p, \xi) \mid |J_1| = \delta_p, |J_2| = \delta_p \right] \in \text{Gr}_2(\mathbb{k}^{\binom{N+\delta_p}{N}}) \subset \mathbb{P}(\Lambda^2 \mathbb{k}^{\binom{N+\delta_p}{N}}).$$

This map Δ^p plays a crucial role in our proof, thus we will now investigate its indeterminacy locus (which is precisely the base locus of the family (β_{J_1,J_2}^p)).

Let us momentarily work again on an affine open set V on which $\mathcal{O}_M(1)$ is trivialized, and for $\xi \in (\pi_{\Omega_M})^{-1}(V)$, consider the evaluation map

$$(5) \quad \eta_{V,\xi}^p: \mathbf{a}^p \mapsto ((\alpha^p)_V(\mathbf{a}^p, \xi), (\theta^p)_V(\mathbf{a}^p, \xi)).$$

It satisfies the following rank condition.

Lemma 2.3. — *For a point $\xi \in (\pi_{\Omega_M})^{-1}(V)$, at which*

$$\#\{i \mid \zeta_i(\xi) = 0\} = (N - k_0) \quad \text{and} \quad \#\{i \mid \zeta_i(\xi) = 0, d\zeta_i(\xi) = 0\} = (N - k_1),$$

for $p = 1, \dots, c$, the rank of the evaluation map $\eta_{V,\xi}^p$ satisfies

$$\text{rank}(\eta_{V,\xi}^p) \geq 2 \binom{k_0 + \delta_p}{k_0} + (k_1 - k_0).$$

Proof. — Since α_J^p and θ_J^p involve only a_J^p , the linear map $\eta_{V,\xi}^p$ is diagonal by blocks. Each block $H^0(M, \mathcal{O}_M(1)) \rightarrow \mathbb{k} \times \mathbb{k}$ corresponds to the map $a_J^p \mapsto ((\alpha_J^p)_V(\mathbf{a}^p, \xi), (\theta_J^p)_V(\mathbf{a}^p, \xi))$ for a certain J . Recall from (2) that locally (for readability, we now drop the “local” notation)

$$\alpha_J^p(\mathbf{a}^p, \xi) = a_J^p(\xi)(\zeta(\xi))^J \text{ and } \theta_J^p(\mathbf{a}^p, \xi) = da_J(\xi)(\zeta(\xi))^J + (r+1)a_J(\xi)d(\zeta)^J(\xi)$$

There is thus two cases, depending on J . Either $(\zeta(\xi))^J \neq 0$, or not.

Before discussing these two cases, let us note that (since sections of very ample line bundles separate first order jets) there exists $b_1 \in H^0(M, \mathcal{O}_M(1))$ such that $b_1(\xi) = 0$ and $db_1(\xi) \neq 0$, and there also exists $b_2 \in H^0(M, \mathcal{O}_M(1))$ such that $b_2(\xi) \neq 0$.

In the first case, when $(\zeta(\xi))^J \neq 0$, it is now immediate that the rank of the block is $= 2$ (i.e. maximal). The respective images $(0, *_{\neq 0})$ and $(*_{\neq 0}, *)$ of b_1 and b_2 cannot be collinear. Thus the rank of the block is 2.

In the other case, when $(\zeta(\xi))^J = 0$, clearly $\alpha_J^p(\mathbf{a}^p, \xi) = a_J^p(\xi)(\zeta(\xi))^J = 0$, whence the rank is at most 1. We consider only particularly simple instances of the this case, namely, we take monomials $\zeta_i^{\delta_p-1} \zeta_j$ with $i \in I := \{i \mid \zeta_i(\xi) \neq 0\}$ and $j \in I' := \{i \mid \zeta_i(\xi) = 0 \text{ and } d\zeta_i(\xi) \neq 0\}$ —it is left to the reader to check that other instances are substantially more complicated—. Then the image of b_2 is $(0, (r+1)b_2(\xi)\zeta_i(\xi)^{\delta_p-1}d\zeta_j(\xi))$. The entry in the second slot is non zero by assumption on i, j . Thus the rank of the block is 1.

By assumption I has cardinality $(k_0 + 1)$ and I' has cardinality $(k_1 - k_0)$. The announced lower bound for the rank follows, after a count of blocks considered in each case. Notice that if $\delta_p \geq 2$ we have even shown

$$\text{rank}(\eta_{V,\xi}^p) \geq 2 \binom{k_0 + \delta_p}{k_0} + (k_0 + 1)(k_1 - k_0). \quad \square$$

This Lemma in turn implies the following.

Proposition 2.4. — *As soon as $\delta_p \geq 2$, there exists a non-empty open subset $U_{\text{def},p} \subset S_p$ such that the indeterminacy locus of Δ^p does not intersect $U_{\text{def},p} \times \mathbb{P}(\Omega_M)$.*

Proof. — Given an affine covering $\mathfrak{B} = (V)$ on which $\mathcal{O}_M(1)$ is trivialized, with finitely many elements, it suffices to show the announced result on each chart. We fix such a chart V .

Recall that we have fixed $\zeta_0, \dots, \zeta_N \in H^0(M, \mathcal{O}_M(1))$ in general position. The total space of the cotangent bundle can thus be stratified by contact orders with the $(N+1)$ hyperplanes $D_i = (\zeta_i = 0)$; For $\emptyset \subsetneq I \subset I' \subset \{0, \dots, N\}$ of respective cardinalities $(1 + k_0)$ and $(1 + k_1)$, we denote by

$$\Sigma(I, I') \subset \mathbb{P}(\Omega_M)$$

the stratum of points such that

$$I = \{0, \dots, N\} \setminus \{i \mid \zeta_i(\xi) = 0\},$$

and

$$I' = \{0, \dots, N\} \setminus \{i \mid \zeta_i(\xi) = 0, d\zeta_i(\xi) = 0\}.$$

It has dimension $(k_0 + k_1 - 1)$.

We will establish that on each stratum $\Sigma(I, I')$, the indeterminacy locus $B \subset S_p \times \Sigma(I, I')$ of the map $(\mathbf{a}^p, \xi) \mapsto \Delta^p(\mathbf{a}^p, \xi)$ cannot dominate S_p , and this will prove the proposition. Accordingly, we fix $I \subset I'$.

By linearity of the determinant, $B = B^\alpha \sqcup (B \setminus B^\alpha)$, where

$$B^\alpha := \{(\mathbf{a}^p, \xi) \in S_p \times \Sigma(I, I') \mid \alpha_J^p(\mathbf{a}^p, \xi) = 0 \ \forall J\}.$$

To prove that B does not dominate S_p , it hence suffices to prove that neither B^α nor $B \setminus B^\alpha$ dominate S_p .

We first prove the first part of this statement, namely that B^α does not dominate S_p . Observe that B^α dominates S_p if and only if

$$B_M^\alpha := \{(\mathbf{a}^p, x) \in S_p \times M \mid \zeta_i(x) \neq 0 \Leftrightarrow i \in I \text{ and } \forall J: \alpha_J^p(\mathbf{a}^p, x) = 0\}$$

does. Fix one x satisfying the conditions on $\zeta_i(x)$. The space of such $x \in M$ has dimension k_0 . One is easily convinced that all $\alpha_J^p(x) = 0$ if and only if, for all J which support is included in I , one has $\alpha_J(x) = 0$. As there are $\binom{k_0 + \delta_p}{k_0}$ such multi-indices J , we infer that

$$\dim(B_M^\alpha) - \dim(S_p) \leq k_0 + \dim(B_{M,x}^\alpha) - \dim(S_p) \leq k_0 - \binom{k_0 + \delta_p}{k_0} \leq \max\{-1, -\delta_p\}.$$

Hence, B_M^α does not dominate S_p .

We then prove the second part of the sought statement, namely that $B \setminus B^\alpha$ does not dominate S_p .

For $\xi \in \Sigma(I, I')$, fix as above a trivialization of the bundle $\mathcal{O}_M(1)$ around $\pi_{\Omega_M}(\xi) \in V$, recall

$$\eta_{V,\xi}^p(\mathbf{a}^p) = ((\alpha^p)_V(\mathbf{a}^p, \xi), (\theta^p)_V(\mathbf{a}^p, \xi)).$$

From the definition of Δ^p , we see that (\mathbf{a}^p, ξ) belongs to the indeterminacy locus of Δ^p if and only if the rank of $\{(\alpha^p)_V(\mathbf{a}^p, \xi), (\theta^p)_V(\mathbf{a}^p, \xi)\}$ is strictly less than 2. Using the stratification, we can even be more precise. Since $\xi \notin B^\alpha$, i.e. $(\alpha^p)_V(\mathbf{a}^p, \xi) \neq 0$, this rank condition means that there exists $\lambda(\mathbf{a}^p, \xi) \in \mathbf{k}$ such that

$$(\theta^p)_V(\mathbf{a}^p, \xi) = \lambda(\mathbf{a}^p, \xi) \cdot (\alpha^p)_V(\mathbf{a}^p, \xi).$$

In particular, $\alpha_J^p(\mathbf{a}^p, \xi) = 0 \Rightarrow \theta_J^p(\mathbf{a}^p, \xi) = 0$, and we can work with multi-indices J with $[J] \subset I$. Since no confusion could arise, we still denote α^p the collection of α_J^p for such indices J . To sum up, we have a map

$$\eta_V^p: \begin{cases} B \setminus B^\alpha \rightarrow \mathbf{k}^{\binom{k_0 + \delta_p}{k_0}} \times \mathbf{k} \\ (\mathbf{a}^p, \xi) \mapsto ((\alpha^p)_V(\mathbf{a}^p, \xi), \lambda(\mathbf{a}^p, \xi)), \end{cases}$$

such that $\eta_{V,\xi}^p = \eta_V^p(\cdot, \xi)$. The dimension of the kernel of this map being precisely $(\dim S_p - \text{rank}(\eta_{V,\xi}^p))$, it follows that

$$\dim(B \setminus B^\alpha) \leq \binom{k_0 + \delta_p}{k_0} + 1 + \dim S_p - \min_{\xi \in \Sigma(I, I')} \text{rank}(\eta_{V,\xi}^p) + \dim(\Sigma(I, I')).$$

This yields, taking account of the different dimensions and of the above lemma

$$\dim(B \setminus B^\alpha) - \dim S_p \leq \binom{k_0 + \delta_p}{k_0} + 1 - 2 \binom{k_0 + \delta_p}{k_0} - (k_1 - k_0) + k_0 + k_1 - 1 \leq 2k_0 - \binom{k_0 + \delta_p}{k_0}.$$

It is left to the reader to check that since $\delta_p \geq 2$, the last expression is < 0 . Therefore $B \setminus B^\alpha$ does not dominate S_p and this finishes the proof. \square

Therefore the restriction $\Delta^p|_{U_{\text{def},p} \times \mathbf{P}(\Omega_M)}$ is a morphism and not merely a rational map. Denote $U_{\text{def}} := U_{\text{def},1} \times \dots \times U_{\text{def},c} \subset S$ and, for the sake of readability, denote

$$\mathbf{G}_2(\delta) := \text{Gr}_2(H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(\delta))),$$

assume $\delta_1, \dots, \delta_c \geq 2$, then we get a map

$$\psi: \begin{cases} U_{\text{def}} \times \mathbf{P}(\Omega_M) \rightarrow \mathbf{G}_2(\delta_1) \times \dots \times \mathbf{G}_2(\delta_c) \times \mathbf{P}^N \\ (\mathbf{a}^\bullet, \xi) \mapsto ([\beta^1(\mathbf{a}^1 \xi)], \dots, [\beta^c(\mathbf{a}^c \xi)], [(\zeta(\xi))^r]). \end{cases}$$

Along $\mathcal{X}^{[1]} \cap (U_{\text{def}} \times \mathbb{P}^N)$, the map \mathcal{P} factors through the universal family

$$\mathcal{Y} := \{(\Delta_1, \dots, \Delta_c, z) \in \mathbf{G} \times \mathbb{P}^N \mid \forall P \in \Delta_1, \dots, \Delta_c: P(z) = 0\},$$

parametrized by $\mathbf{G} := \mathbf{G}_2(\delta_1) \times \dots \times \mathbf{G}_2(\delta_c)$. Since $2c \geq N$ the projection $\rho: \mathcal{Y} \rightarrow \mathbf{G}$ is generically finite, and might be not surjective.

Formally, if one forgets that $N - 2c$ could be negative, one can think of \mathcal{Y} as the universal family of complete intersections of codimension $N - 2c$ and multi-degree $(\delta_1, \delta_1, \dots, \delta_c, \delta_c)$ parametrized by \mathbf{G} . Keeping this in mind, thanks a result of Benoist (see Section 2.3 below), we obtain a key ingredient needed in our proof, namely the fact that the locus of elements in \mathbf{G} that parametrize positive dimensional schemes is of large codimension in \mathbf{G} .

Before proceeding, for technical reasons, we need to introduce the stratification of $\mathcal{X}^{[1]}$ induced by the vanishing of the sections ζ_0, \dots, ζ_N on the base M . Namely, for each $I \subseteq \{0, \dots, N\}$ with $\#I = k + 1$, let

$$M_I := \{x \in M \mid \zeta_i(x) \neq 0 \Leftrightarrow i \in I\},$$

be the k dimensional subspace of M associated to I , pullback this stratification to $\mathbb{P}(\Omega_M)$, obtaining $\mathbb{P}(\Omega_M|_{M_I}) := (\pi_{\Omega_M})^{-1}(M_I)$, and take the intersection with $\mathcal{X}^{[1]}$. For reasons that will soon become obvious, we are mainly interested in the case where $\dim(M_I) = k \geq 1$.

Fact 2.5. — *Let $I = (i_0, \dots, i_k)$ with $k \geq 1$. Along $\mathcal{X}^{[1]} \cap (U_{\text{def}} \times \mathbb{P}(\Omega_M|_{M_I}))$, the map \mathcal{P} factors through*

$$\mathcal{Y}(I) := \{(\Delta_1, \dots, \Delta_c, z) \in \mathbf{G} \times \mathbb{P}(k^I) \mid \forall P \in \Delta_1, \dots, \Delta_c: P(z) = 0\}.$$

Proof. — Locally, on an affine open subset $V \subset M$, the variety $\mathcal{X}^{[1]} \cap (S \times V)$ is defined precisely by the equations $(E^p)_V = \sum (\alpha_j^p)_V (\zeta_V)^{r_j}$ and $(dE^p)_V = \sum (\theta_j^p)_V (\zeta_V)^{r_j}$. A straightforward computation then proves that $\mathcal{P}|_{\mathcal{X}^{[1]}}$ factors through \mathcal{Y} . Moreover, by definition, as $\xi \in \mathbb{P}(\Omega_M|_{M_I}) = \pi_{\Omega_M}^{-1} M_I$ we obtain $(\zeta(\xi))^r \in \mathbb{P}(k^I) \subset \mathbb{P}^N$. \square

2.3. Avoiding the exceptional locus. — Because we want to use the positivity of the base \mathbf{G} , it is important to avoid positive dimensional fibers of $\mathcal{Y}(I) \rightarrow \mathbf{G}$. We denote the *exceptional locus* of a generically finite morphism $f: X \rightarrow Y$ between two varieties by $\text{Exc}(f) := \{x \in X \mid \dim_x(f^{-1}(\{f(x)\})) > 0\}$.

Lemma 2.6. — *If $\delta_1, \dots, \delta_c \geq \dim(\mathbb{P}(\Omega_M))$, then for each $I = (i_0, \dots, i_k) \subset \{0, \dots, N\}$ with $k \geq 1$, there exists a non-empty open subset $U_I \subset S$ such that*

$$\mathcal{P}(\mathcal{X}^{[1]} \cap (U_I \times \mathbb{P}(\Omega_M|_{M_I}))) \cap \text{Exc}(\mathcal{Y}(I) \rightarrow \mathbf{G}) = \emptyset.$$

Proof. — For such I , for $\xi \in \mathbb{P}(\Omega_M|_{M_I})$, and for an affine neighborhood $V \ni \pi_{\Omega_M}(\xi)$ in which $\mathcal{O}_M(1)$ is trivial, the maps $\eta_{V,\xi}^p: \mathbf{a}^p \mapsto (\alpha^p(\mathbf{a}^p, \xi), \theta^p(\mathbf{a}^p, \xi))$ already discussed above induce a morphism $\hat{\mathcal{P}}: \mathcal{X}^{[1]} \cap (U_{\text{def}} \times \mathbb{P}(\Omega_M|_{M_I})) \rightarrow \hat{\mathcal{Y}}(I)$ into an affine analogue $\hat{\mathcal{Y}}(I)$ of the universal family $\mathcal{Y}(I)$, namely

$$\hat{\mathcal{Y}}(I) := \{(\alpha_1, \theta_1, \dots, \alpha_c, \theta_c, z) \in \mathbf{A} \times \mathbb{P}(k^I) \mid \alpha_i(z) = 0, \theta_i(z) = 0 \forall 1 \leq i \leq c\},$$

where $\mathbf{A} := \bigoplus_{p=1}^c H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta_p))$, is the natural affine analogue of $\mathbf{G}(I)$. The restriction map $p: \mathbb{P}^N \rightarrow \mathbb{P}(k^I)$ induces a map $p: \mathbf{A} \rightarrow \mathbf{A}(I) := \bigoplus_{p=1}^c H^0(\mathbb{P}(k^I), \mathcal{O}_{\mathbb{P}(k^I)}(\delta_p))$. Consider the variety

$$\tilde{\mathcal{Y}}(I) := \{(\alpha_1, \theta_1, \dots, \alpha_c, \theta_c, z) \in \mathbf{A}(I) \times \mathbb{P}(k^I) \mid \alpha_i(z) = 0, \theta_i(z) = 0 \forall 1 \leq i \leq c\}.$$

For shortness, let us denote

$$E := \text{Exc}(\rho: \mathcal{Y}(I) \rightarrow \mathbf{G}), \quad \hat{E} := \text{Exc}(\hat{\rho}: \hat{\mathcal{Y}}(I) \rightarrow \mathbf{A}), \quad \tilde{E} := \text{Exc}(\tilde{\rho}: \tilde{\mathcal{Y}}(I) \rightarrow \mathbf{A}(I)).$$

Then we define the “non-finite” loci $\mathbf{G}^\infty = \rho(E)$, $\mathbf{A}^\infty = \hat{\rho}(\hat{E})$, and $\mathbf{A}^\infty(I) = \tilde{\rho}(\tilde{E})$. A straightforward computation shows that

$$\mathbf{A}^\infty = p^{-1}(\mathbf{A}^\infty(I)).$$

The framework being set-up, let us count codimension. Fix ξ as above. Since we work locally, the map

$$\hat{\Phi}_{V,\xi}: \mathbf{a}^\bullet \mapsto (\eta_{V,\xi}^1(\mathbf{a}^1), \dots, \eta_{V,\xi}^c(\mathbf{a}^c))$$

is well-defined. Note that

$$\text{pr}_1^{[1]}(\mathcal{X}^{[1]} \cap (U_{\text{def}} \times (\text{pr}_2^{[1]})^{-1}(\xi)) \cap \mathcal{P}^{-1}(E) \subseteq \Phi_{V,\xi}^{-1}(\mathbf{A}^\infty) \subseteq \Phi_{V,\xi}^{-1}(p^{-1}(\mathbf{A}^\infty(I))).$$

Moreover, just as in Lemma 2.3, one proves that

$$\text{rank}(p \circ \hat{\Phi}_{V,\xi}) = \sum_{p=1}^c 2 \binom{k + \delta_p}{k} = \dim \mathbf{A}(I).$$

Hence,

$$\dim(\mathcal{X}^{[1]} \cap (\mathrm{pr}_2^{[1]})^{-1}(\xi) \cap \Psi^{-1}(E)) \leq \dim((p \circ \Phi_{V,\xi})^{-1}(\mathbf{A}^\infty(I))) \leq \dim S - \dim \mathbf{A}(I) + \dim \mathbf{A}^\infty(I).$$

This yields

$$\dim(\mathcal{X}^{[1]} \cap (U_{\mathrm{def}} \times \mathbb{P}(\Omega_M|_{M_I})) \cap \Psi^{-1}(E)) - \dim S \leq \dim \mathbb{P}(\Omega_M|_{M_I}) - \mathrm{codim}(\mathbf{A}^\infty(I), \mathbf{A}(I)).$$

Now, by Corollary 3.2, and by the hypothesis of the theorem

$$\mathrm{codim}(\mathbf{A}^\infty, \mathbf{A}(I)) = \mathrm{codim}(\mathbf{G}^\infty, \mathbf{G}(I)) \stackrel{3.2}{\geq} \min \delta_p + 1 \stackrel{\mathrm{hyp.}}{\geq} \dim(\mathbb{P}(\Omega_M)) + 1 > \dim(\mathbb{P}(\Omega_M|_{M_I})),$$

whence

$$\dim(\mathcal{X}^{[1]} \cap (U_{\mathrm{def}} \times \mathbb{P}(\Omega_M|_{M_I})) \cap \Psi^{-1}(E)) - \dim S < 0.$$

This dimension count shows that $\mathcal{X}^{[1]} \cap (U_{\mathrm{def}} \times \mathbb{P}(\Omega_M|_{M_I})) \cap \Psi^{-1}(E)$ cannot dominate S . \square

2.4. Pulling back the positivity. — We are now in position to implement the last part of our strategy, thereby establishing the following stronger version of Theorem 1.1.

Theorem 2.7. — *For each choice of $\delta_1, \dots, \delta_c \geq \dim \mathbb{P}(\Omega_M)$, there exists an integer $m(\delta)$ such that for any $r > 2m(\delta)(|\varepsilon| + |\delta|)$, the general member of the family $\mathcal{X} = \mathcal{X}(\delta, \varepsilon, r)$ constructed in Section 2.1 has ample cotangent bundle.*

By the openness property of ampleness, this immediately implies the following.

Corollary 2.8. — *On a N -dimensional smooth projective variety M , equipped with a very ample line bundle $\mathcal{O}_M(1)$, for codimensions $N/2 \leq c \leq N$, for any degrees $(d_1, \dots, d_c) \in (\mathbb{N}^*)^c$ satisfying*

$$(\dagger) \quad \exists \delta_{(p \geq \dim(\mathbb{P}(\Omega_M)))} \exists \varepsilon_{(\varepsilon_p \geq 1)}, \exists r > 2m(\delta)(|\varepsilon| + |\delta|): \quad d_p = \delta_p(r+1) + \varepsilon_p \quad (p = 1, \dots, c),$$

the complete intersection $X := H_1 \cap \dots \cap H_c$ of general hypersurfaces $H_1 \in |\mathcal{O}(d_1)|, \dots, H_c \in |\mathcal{O}(d_c)|$ has ample cotangent bundle.

Condition (\dagger) will be discussed in Section 2.5 below.

Proof of Theorem 2.7. — It is sufficient to prove that $\mathcal{O}_{\mathbb{P}(\Omega_{X_{a^\bullet}}(-1/\mu))}(\mu) = \mathcal{O}_{\mathbb{P}(\Omega_{X_{a^\bullet}})}(\mu) \otimes (\pi_{a^\bullet})^* \mathcal{O}_{X_{a^\bullet}}(-1)$ is nef for a sufficiently large μ and a^\bullet general, where we denote $\pi_{a^\bullet} : \mathbb{P}(\Omega_{X_{a^\bullet}}) \rightarrow X_{a^\bullet}$. For such μ and a^\bullet , we have thus to prove that any irreducible curve $\mathcal{C} \subset \mathbb{P}(\Omega_{X_{a^\bullet}})$ satisfies

$$(*) \quad \mathcal{C} \cdot (\mathcal{O}_{\Omega_{X_{a^\bullet}}}(\mu) \otimes (\pi_{a^\bullet})^* \mathcal{O}_{X_{a^\bullet}}(-1)) \geq 0.$$

Observe that, since $\mathbb{P}(\Omega_M|_{M_I})$ stratifies $\mathbb{P}(\Omega_M)$, for each irreducible curve $\mathcal{C} \subset \mathbb{P}(\Omega_{X_{a^\bullet}}) \subset \mathbb{P}(\Omega_M)$, there exists a unique set $I = (i_0, \dots, i_k)$ such that $\mathbb{P}(\Omega_M|_{M_I})$ contains an open dense subset of \mathcal{C} .

We fix such a curve \mathcal{C} , and the corresponding I . If $\dim(M_I) = k = 0$, $(*)$ holds for all μ and a^\bullet . If in the contrary $k \geq 1$, with above notation, we work with the tautological ample line bundle

$$Q := \mathcal{O}_{\mathbb{G}_2(\delta_1)}(1) \boxtimes \dots \boxtimes \mathcal{O}_{\mathbb{G}_2(\delta_c)}(1) \longrightarrow \mathbf{G} = \mathbf{G}_2(\delta_1) \times \dots \times \mathbf{G}_2(I, \delta_c).$$

Since Q is ample and $\rho_I : \mathcal{Y}(I) \rightarrow \mathbf{G}$ generically finite, the pullback-bundle $\rho_I^* Q$ is big and nef. Accordingly, its *non-ample locus*—which, thanks to a theorem due to Nakamaye, is known to coincide with the *augmented base locus* $\mathbf{B}_+(\rho_I^* Q)$, see [Nak00] or e.g. [Laz04] Section 10.3—is exactly the reunion of all positive dimensional fibers of ρ_I , i.e. the exceptional locus of the map $\mathcal{Y}_I \rightarrow \mathbf{G}$. From the very definition of \mathbf{B}_+ , the fact that $Q \boxtimes \mathcal{O}_{\mathbb{P}(k^I)}(1)$ is ample, and Noetherianity, we know that there exists an integer $m_I(\delta)$ such that for $m \geq m_I(\delta)$,

$$\mathbf{B}_+(\rho_I^* Q) = \mathrm{Bs}((Q^{\otimes m} \boxtimes \mathcal{O}_{\mathbb{P}(k^I)}(-1))|_{\mathcal{Y}(I)}).$$

To sum up, there exists $m_I(\delta)$ such that for $m \geq m_I(\delta)$

$$E(I) := \mathrm{Exc}(\mathcal{Y}(I) \rightarrow \mathbf{G}) = \mathrm{Bs}((Q^{\otimes m} \boxtimes \mathcal{O}_{\mathbb{P}(k^I)}(-1))|_{\mathcal{Y}(I)}).$$

In order to work uniformly on all $\mathbb{P}(\Omega_M|_{M_I})$, set

$$m(\delta) := \max \{m_I(\delta) \mid I \subseteq \{0, \dots, N\}, \#I \geq 2\}.$$

We will now fix μ . From the definition of the map Δ^p , we have $\Delta^p|_{U_{\mathrm{def}}}^* \mathcal{O}_{\mathbb{G}_2(\delta_p)}(1) = \mathbb{L}(2(\varepsilon_p + \delta_p))$. It follows at once from the definition of Ψ and from (4) that

$$(*_2) \quad \Psi|_{\mathcal{X}^{[1]} \cap (U_{\mathrm{def}} \times \mathbb{P}(\Omega_M|_{M_I}))}^* (Q^{\otimes m(\delta)} \boxtimes \mathcal{O}_{\mathbb{P}(k^I)}(-1)) = \mathcal{O}_{\mathbb{P}(\Omega_{X/S})}(cm(\delta)) \otimes \pi_{X/S}^* \mathcal{O}_X(-r + 2m(\delta)(|\varepsilon| + |\delta|)).$$

We take $\mu := cm(\delta)$. Observe the $-r$; with the assumption on r in the theorem the above bundle becomes a negative twist of the bundle of symmetric differential forms.

Then, we proceed as follows. Since $\delta_1, \dots, \delta_c \geq \dim(\mathbb{P}(\Omega_M))$, the conclusion of Lemma 2.6 is satisfied for any r . Hence for $\mathbf{a}^\bullet \in U(I)$, $\Psi(\mathbb{P}(\Omega_{X_{\mathbf{a}^\bullet}}) \cap \mathbb{P}(\Omega_{M|_{M_I}})) \cap E(I) = \emptyset$. Therefore,

$$\mathcal{C} \not\subset \Psi^{-1}(E(I)) = \Psi^{-1} \text{Bs}(Q^{\otimes m(\delta)} \boxtimes \mathcal{O}_{\mathbb{P}^k}(-1)|_{\mathcal{Y}(I)}).$$

This implies the existence of a section $\sigma \in H^0(\mathcal{Y}(I), Q^{\otimes m(\delta)} \boxtimes \mathcal{O}_{\mathbb{P}^k}(-1)|_{\mathcal{Y}(I)})$ such that $\mathcal{C} \not\subset (\Psi^*\sigma = 0)$. In particular, $\mathcal{C} \cdot (\Psi^*\sigma = 0) \geq 0$. Therefore by $(*)_2$,

$$\mathcal{C} \cdot (O_{\Omega_{X_{\mathbf{a}^\bullet}}}(\mu) \otimes (\pi_{\mathbf{a}^\bullet})^* O_{X_{\mathbf{a}^\bullet}}(-r + m(\delta)(|\varepsilon| + |\delta|))) \geq 0.$$

This proves $(*)$ for any curve in any stratum, as soon as $\mathbf{a}^\bullet \in \cap_I U(I)$, and thus finishes the proof. \square

2.5. The condition on the degrees. — Up to taking a multiple, one can assume that all δ_i in Theorem 1.1 are larger than $2N - 1$. Then by Corollary 2.8, the degrees d_i for which the conclusion of Theorem 1.1 holds in the direction δ , are the ones satisfying the following condition.

$$(\dagger) \quad \exists \varepsilon_{(p \geq 1)}, \exists r > 2m(\delta)(|\varepsilon| + |\delta|): \quad d_p = \delta_p(r + 1) + \varepsilon_p \quad (p = 1, \dots, c).$$

The picture below shows particular solutions, with $1 \leq \varepsilon_p \leq \delta_p$, of (\dagger) in an arbitrary direction $\delta = (\delta_1, \dots, \delta_c)$, with $\delta_p \geq 2N - 1$.

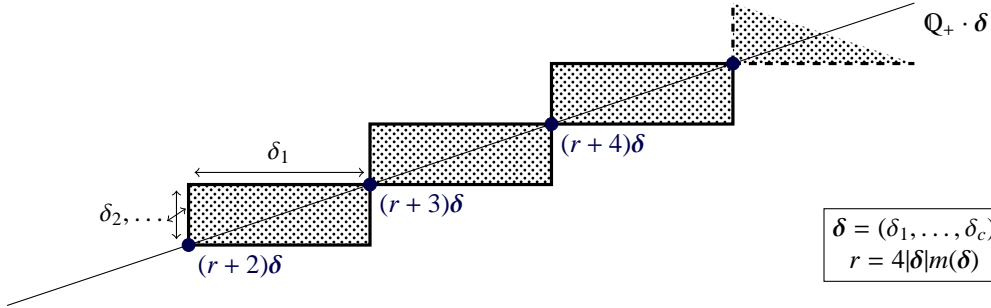


FIGURE 1. Some degrees satisfying the condition (\dagger)

As one can see, these solutions lie in a union of rectangular blocks that contains the ray spanned by δ starting from a sufficiently large multiple $\nu(\delta)\delta$ with e.g. $\nu(\delta) = 4|\delta|m(\delta) + 2$. So, in particular, we treat all large degrees $(d_1, \dots, d_c) = \nu(\delta_1, \dots, \delta_c)$, with a lower bound depending on the direction $\mathbb{Q}_+ \cdot (\delta_1, \dots, \delta_c)$.

3. On the codimension of the exceptional locus

If one parametrizes complete intersections in \mathbb{P}^N by products of spaces of homogeneous polynomials, the locus of families of polynomials not parametrizing complete intersections is “small” compared to the entire parameter space. This follows easily from the work of Benoist ([Ben11]), but since it is playing a central role in our proof, we provide all the details here. We do not claim any originality in this section.

Lemma 3.1 ([Ben11, Lemme 2.3]). — *Let $X \subset \mathbb{P}^N$ be a (irreducible) subvariety dimension n . Let \mathcal{G} be the set of all hypersurfaces of degree e containing X . Then*

$$\text{codim}(\mathcal{G}, |\mathcal{O}_{\mathbb{P}^N}(e)|) \geq \binom{e+n}{n}.$$

Proof. — This is the proof found in [Ben11], we translate it here for the reader’s convenience. Let L be a $(N - n - 1)$ -dimensional linear subspace of \mathbb{P}^N such that $L \cap X = \emptyset$. Let $\pi_L : X \rightarrow \mathbb{P}^n$ be the projection induced by L . All the fibers of π_L are non-empty and finite. This proves that if C is the set of all cones of degree e with vertex L , $C \cap \mathcal{G} = \emptyset$, and therefore

$$\text{codim}(\mathcal{G}, |\mathcal{O}_{\mathbb{P}^N}(e)|) \geq \dim C + 1 = \binom{n+e}{n}. \quad \square$$

From this it is straightforward to deduce the required result on complete intersection varieties. For $\gamma \geq 1$ and integers e_1, \dots, e_γ write $T_e := \prod_{i=1}^\gamma |\mathcal{O}_{\mathbb{P}^N}(e_i)|$ and T_e^∞ to be the subvariety of T_e parametrizing equations that do not define a complete intersection variety if $\gamma \leq N$, or to be the subvariety parametrizing equations that do not define a 0-dimensional scheme if $\gamma \geq N$. In the case $\gamma = N$ the two notions coincide. In any case T_e^∞ is closed in T_e . Then we have the following.

Corollary 3.2. — *With the above notation,*

$$\mathrm{codim}(T_e^\infty, T_e) \geq \min_{1 \leq i \leq \min\{N, \gamma\}} \binom{e_i + N - i + 1}{N - i + 1}.$$

In particular, this codimension tends to infinity as the e_i 's tend to infinity, and we have the rough bound $\mathrm{codim}(T_e^\infty, T_e) \geq \min_{1 \leq i \leq \gamma} (e_i + 1)$.

Proof. — The proof is an induction on γ . If $\gamma = 1$, it is obvious. Suppose the result holds for $\gamma < N$. Then for an element $F_1, \dots, F_{\gamma+1} \in T_e$, the subscheme $X_{\gamma+1}$ defined by $F_1, \dots, F_{\gamma+1}$ fails to be a complete intersection, either when the subscheme X_γ defined by the first γ equations fails to be a complete intersection, either when the hypersurface $(F_{\gamma+1} = 0)$ contains one of the irreducible components of X_γ . The codimension of the subset defined by the first condition is greater than $\min_{1 \leq i \leq \gamma} \binom{e_i + N - i + 1}{N - i + 1}$ by induction hypothesis. The codimension of the subset defined by the second condition is greater than $\binom{e_{\gamma+1} + N - \gamma}{N - \gamma}$. This is seen simply by fixing some X_γ which is a complete intersection, so that all irreducible components of X_γ are of dimension $N - \gamma$ and applying Benoist's Lemma to each of those irreducible components. This proves the statement for all $1 \leq \gamma \leq N$, the statement for $\gamma > N$ follows at once. Indeed, if $\gamma > N$ then the codimension of T_e^∞ in T_e is bigger than the codimension of the set of elements in T_e whose N first equations define a finite subscheme of \mathbb{P}^N , and we are reduced to the statement for $\gamma = N$. \square

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